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Theorems of The Alternative for Nearly Convexlike Set-Valued Maps*

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Abstract: In this paper, we obtain some theorems of the alternative for nearly convexlike and nearly subconvexlike set-valued maps in linear topological spaces were obtained by the authors.

Key words: Theorems of the alternative; set-valued maps; near convexlikeness; near subconvexlikeness; linear topological spaces

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Theorems of the alternative play an important role in vector optimization theory, and it becomes an essential tool to study optimality conditions, Lagrange multipliers, duality, ect. Recently, since vector optimization of set-valued maps has received an increasing amount of attention, it is necessary to study the theorems of the alternative for set-valued maps under the assumption of generalized convexity. In Ref. 2, Li established a theorem of the alternative for subconvexlike set-valued maps in ordered linear topological space.

In this paper, we establish some theorems of the alternative for nearly convexlike and nearly subconvexlike set-valued maps by using the separation theorem of convex set in linear topological space. The proof given here is substantially different from that given by Illes and Kassay when vector-valued maps are considered. Our results are also the foundation for deeply discussing vector optimization theory for set-valued maps.

1 Notations and Preliminaries

Let D be an arbitrarily chosen nonempty abstract set. Let Y be a linear topological space. The convex cone Y_+ with apex at the origin in Y is called a positive cone of Y. Suppose that the positive cone Y_+ is not equal to Y; Let A be a nonempty subset of Y. We denote by *int* A the interior of A.

We denote by Y^* the dual of Y. Setting $Y_+^* = \{y^* \in Y^* \mid \langle y, y^* \rangle \geqslant 0, \forall\, y \in Y_+\}$, where $\langle y, y^* \rangle = y^*(y)$; Y_+^* is said to be the dual cone of the positive cone Y_+.

Suppose that $F: D \to 2^Y$ is a set-valued map form D to Y, where 2^Y denotes the power set of Y. Let $F(D) = \bigcup_{x\in D} F(x)$, $\langle F(x), y^* \rangle = \{\langle y, y^* \rangle \mid y \in F(x)\}$, $\langle F(D), y^* \rangle = \bigcup_{x \in D} \langle F(x), y^* \rangle$. For $x \in D$, $y^* \in Y^*$, write $\langle F(x), y^* \rangle \geqslant 0$, iff $\langle y, y^* \rangle \geqslant 0, \forall\, y \in F(x)$; $\langle F(D), y^* \rangle \geqslant 0$, iff $\langle F(x), y* \rangle \geqslant 0, \forall\, x \in D$;

We denote by R set of real numbers. For $A \subset R$, $b \in R$, write $A \geqslant b$, iff $a \geqslant b$, $\forall\, a \in A$.

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Definition 1.1 The subset M of Y is called nearly convex, if $\exists\ \alpha\in(0,1)$, $\forall\ y_1, y_2\in M$, such that $\alpha y_1+(1-\alpha)y_2\in M$.

Definition 1.2 The set-valued map $f:D\to 2^Y$ is called nearly convexlike, if $\exists\ \alpha\in(0,1)$, $\forall\ x_1, x_2\in D$, $\exists\ z\in D$, such that

$$\alpha f(x_1)+(1-\alpha)f(x_2)\subset f(z)+Y_+$$

Lemma 1.1 (See Ref. 1) If $M\subset Y$ is a nearly convex set, then *int* M is a convex set.

Lemma 1.2 (See Ref. 2) Let $Y_+\subset Y$ be a positive cone, and let *int* Y_+ be nonempty. Suppose that Y_+^* is the dual cone of Y_+. If $y_0^*\in Y_+^*$, $y_0^*\neq O_{Y^*}$, $y_0\in int\ Y_+$, then $\langle y_0,y_0^*\rangle>0$.

2 Theorems of the Alternative

In the following, we consider two linear topological spaces Y_1 and Y_2. Suppose that Y_{i+} is the positive cone of Y_i, $i=1,2$; Suppose that *int* Y_{1+} is nonempty. However, the interior of Y_{2+} is not required to be nonempty. Let Y_{i+}^* is the dual cone of Y_{i+}, $i=1,2$.

Let $f:D\to 2^{Y_1}$, $g:D\to 2^{Y_2}$. Put $Y=Y_1\times Y_2$, $Y_+=Y_{1+}\times Y_{2+}$, $F=(f,g):D\to 2^Y$. Obviously, $Y^*=Y_1^*\times Y_2^*$, and one can easily verify that $Y_+^*=Y_{1+}^*\times Y_{2+}^*$.

Lemma 2.1 If $y_0^*=(y_{01}^*, y_{02}^*)\in Y_+^*$, $y_{01}^*\neq O_{Y_1^*}$, $y_0=(y_{01}, y_{02})\in(int\ Y_{1+})\times Y_{2+}$, then $\langle y_0, y_0^*\rangle>0$.

Proof Since $y_{01}\in int\ Y_{1+}$, it follows by Lemma 1.2 that $\langle y_{01},y_{01}^*\rangle>0$. By definition of Y_{2+}^*, we get $\langle y_{02},y_{02}^*\rangle\geqslant 0$. So,

$$\langle y_0,y_0^*\rangle=\langle y_{01},y_{01}^*\rangle+\langle y_{02},y_{02}^*\rangle>0$$

The proof is complete.

Lemma 2.2 $int\ (F(D)+Y_+)\neq\varnothing$, if and only if $int(F(D)+(int\ Y_{1+})\times Y_{2+})\neq\varnothing$.

Proof The proof is similar to the proof of Lemma 2.2 of Ref. 1 when F is a vector-valued map.

Lemma 2.3 Let $M=F(D)+(int\ Y_{1+})\times Y_{2+}$. If F is nearly convexlike, then M is nearly convex.

Proof Let $c_1,c_2\in M$. Then, there exist $x_i\in M$, $y_{i1}\in int\ Y_{1+}$, $y_{i2}\in Y_{2+}$, $i=1,2$ satisfying

$$c_1\in F(x_1)+(y_{11},y_{12}),\quad c_2\in F(x_2)+(y_{21},y_{22})$$

Hence, $\alpha c_1+(1-\alpha)c_2\in\alpha F(x_1)+(1-\alpha)F(x_2)+\alpha(y_{11},y_{12})+(1-\alpha)(y_{21},y_{22})$, $\forall\ \alpha\in(0,1)$.
Since $(int\ Y_{1+})\times Y_{2+}$ is convex, then $\alpha(y_{11},y_{12})+(1-\alpha)(y_{21},y_{22})\in(int\ Y_{1+})\times Y_{2+}$. Therefore,

$$\alpha c_1+(1-\alpha)c_2\in\alpha F(x_1)+(1-\alpha)F(x_2)+(int\ Y_{1+})\times Y_{2+}, \forall\ \alpha\in(0,1) \tag{1}$$

Due to the assumption that F is nearly convexlike, consequently $\exists\ \alpha_0\in(0,1)$, $\forall\ x_1,x_2\in D$, $\exists\ z\in D$, such that

$$\alpha_0 F(x_1)+(1-\alpha_0)F(x_2)\subset F(z)+Y_+ \tag{2}$$

Setting $\alpha=\alpha_0$ in (1), we get

$$\alpha_0 c_1+(1-\alpha_0)c_2\in\alpha_0 F(x_1)+(1-\alpha_0)F(x_2)+(int\ Y_{1+})\times Y_{2+} \tag{3}$$

Hence, it follows by (2), (3) that $\alpha_0 c_1+(1-\alpha_0)c_2\in F(z)+Y_++(int\ Y_{1+})\times Y_{2+}$.
Because of $Y_{1+}+int\ Y_{1+}\subset int\ Y_{1+}$, we obtain $Y_++(int\ Y_{1+})\times Y_{2+}\subset(int\ Y_{1+})\times Y_{2+}$. So,

$$\alpha_0 c_1+(1-\alpha_0)c_2\in F(z)+(int\ Y_{1+})\times Y_{2+}\subset F(D)+(int\ Y_{1+})\times Y_{2+}$$

The proof of near convexity of M is complete.

Lemma 2.4 (See Ref. 1) Let $M=F(D)+(int\ Y_{1+})\times Y_{2+}$. Suppose that M is nearly convex, and suppose that the interior of M is nonempty. If $\exists\ y^*=(y_1^*,y_2^*)\in Y_1^*\times Y_2^*$, $y^*\neq O_{Y^*}$, such that

$\langle u,y^*\rangle=\langle u_1,y_1^*\rangle+\langle u_2,y_2^*\rangle>0$, $\forall\, u\in int\, M$. Then

$$\langle u,y^*\rangle\geqslant 0,\ \forall\, u\in M$$

In the following, we consider the following generalized inequality-equality systems:

System 1 $\exists\, x_0\in D, s.t.\ -f(x_0)\cap int\, Y_{1+}\neq\varnothing, -g(x_0)\cap Y_{2+}\neq\varnothing$;

System 2 $\exists\, y^*=(y_1^*,y_2^*)\in Y_{1+}^*\times Y_{2+}^*,\ y^*\neq O_{Y^*}, s.t.$

$$\langle f(x),y_1^*\rangle+\langle g(x),y_2^*\rangle\geqslant 0,\ \forall\, x\in D \tag{4}$$

For the above two systems, we have the following alternative theorem which generalized Theorem 3.1 of Ref. 1, the Farkas-Minkowski type theorem for vector-valued maps.

Theorem 2.1 (Alternative Theorem) Suppose that the set-valued map $F=(f,g):D\to 2^Y$ is nearly convexlike, and suppose that the interior of $F(D)+Y_+$ is nonempty.

1) If System 2 has a solution (y_1^*,y_2^*) with $y_1^*\neq O_{Y_1^*}$, then System 1 has no solution;

2) If System 1 has no solution, then System 2 has a solution (y_1^*,y_2^*).

Proof

1) Suppose that System 2 has a solution (y_1^*,y_2^*) with $y_1^*\neq O_{Y_1^*}$. If System 1 has a solution $x_0\in D$, there exist $p\in f(x_0)$, $q\in g(x_0)$, such that $-p\in int\, Y_{1+}$, $-q\in Y_{2+}$. Thus, by Lemma 2.1, we get $\langle p,y_1^*\rangle+\langle q,y_2^*\rangle<0$. This is a contradiction to (4).

2) Since F is nearly convexlike, $M=F(D)+(int\, Y_{1+})\times Y_{2+}$ is nearly convex. Therefore, $int\, M$ is convex. By the assumption that $int\,(F(D)+Y_+)$ is nonempty, we get $int\, M\neq\varnothing$.

Since System 1 has no solution, $O_Y=(O_{Y_1},O_{Y_2})\notin M$. Hence, by using the separation theorem of convex sets of linear topological spaces (See Ref 3), there exists a hyperplane H properly separating $\{O_Y\}$ and $int\, M$. *i.e.*, $\exists\, y^*=(y_1^*,y_2^*)\in Y_1^*\times Y_2^*, y^*\neq O_{Y^*}, a\in R$, such that

$$\langle u,y^*\rangle\geqslant a\geqslant 0,\quad \forall\, u\in int\, M \tag{5}$$

where the hyperplane $H=\{y\in Y\,|\,\langle y,y^*\rangle=a\}$.

Now, we show that

$$\langle u,y^*\rangle>0,\quad \forall\, u\in \text{int}\, M \tag{6}$$

If $a>0$, it follows by (5) that (6) holds. Assume that $a=0$. Also by (5), we obtain

$$\langle u,y^*\rangle\geqslant 0,\quad \forall\, u\in \text{int}\, M. \tag{7}$$

Suppose that (6) does not hold. Then, by (7), there exists $u_0\in int\, M$ such that

$$\langle u_0,y^*\rangle=0 \tag{8}$$

Because of $u_0\in int\, M$, there exists a neighborhood N of the origin O_Y such that $u_0+N\in int\, M$. Due to the fact that N is absorbent, we may select a positive number ε which is sufficiently small such that $-\varepsilon v\in N$, $\forall\, v\in int\, M$. Therefore, $u_0-\varepsilon v\in int\, M$. By (7), we get $\langle u_0-\varepsilon v,\ y^*\rangle\geqslant 0$. *i.e.*, $\langle u_0,y^*\rangle\geqslant\langle\varepsilon v,y^*\rangle$. By (8), we obtain $\langle v,y^*\rangle\leqslant 0$. Since $v\in int\, M$, by (7), we have $\langle v,y^*\rangle\geqslant 0$. Therefore, $\langle u,y^*\rangle=0$, $\forall\, v\in int\, M$. This implies that the hyperplane H cannot properly separate $\{O_Y\}$ and $int\, M$, which contradicts the separation theorem. Thus, the proof that (6) holds is complete.

By Lemma 2.4, we obtain

$$\langle u,y^*\rangle\geqslant 0,\quad \forall\, u\in M \tag{9}$$

Next, we show that $(y_1^*,y_2^*)\in Y_{1+}^*\times Y_{2+}^*$. In faet, assume that $y_1^*\notin Y_{1+}^*$. There exists $y_1\in Y_{1+}$ such that $\langle y_1,y_1^*\rangle<0$. Then, $\lambda\langle y_1,y_1^*\rangle=\langle\lambda y_1,y_1^*\rangle<0, \forall\,\lambda>0$. By (9), for any $x\in D$, $y'_1\in int\, Y_{1+}$, $y_2'\in Y_{2+}$, we have $\beta=\langle p+y'_1,y_1^*\rangle+\langle q+y'_2,y_2^*\rangle\geqslant 0, \forall\, p\in f(x), q\in g(x)$. Since $\lambda y_1\in Y_{1+}$, hence $\lambda y_1+y'_1\in int\, Y_{1+}$. Also by (8), we get $\langle p+\lambda y_1+y'_1,y_1^*\rangle+\langle q+y'_2,y_2^*\rangle\geqslant 0$. *i.e.*,

$$\lambda\langle y_1, y_1^*\rangle + \beta \geqslant 0, \quad \forall \lambda > 0 \tag{10}$$

However, (10) does not hold when $\lambda > -\beta/\langle y_1, y_1^*\rangle$, since $-\beta/\langle y_1, y_1^*\rangle \geqslant 0$. This contradiction illustrates that $y_1^* \in Y_{1+}^*$.

The proof of $y_2^* \in Y_{2+}^*$ is similar to the proof of $y_1^* \in Y_{1+}^*$. Therefore, $\exists\ y^* = (y_1^*, y_2^*) \in Y_{1+}^* \times Y_{2+}^*$, $y^* \neq O_{Y^*}$, s. t. $\langle u, y^*\rangle \geqslant 0, \forall\ u \in M$. i. e.,

$$\langle F(x) + t, y^*\rangle \geqslant 0, \quad \forall\ x \in D,\ t \in (int\ Y_{1+}) \times Y_{2+}$$

Take $t_0 \in (int\ Y_{1+}) \times Y_{2+}$ and $\lambda_n > 0$ such that $\lambda_n \to 0$ for $n \to +\infty$; then, letting $n \to +\infty$, we have

$$\langle F(x), y^*\rangle = \langle f(x), y_1^*\rangle + \langle g(x), y_2^*\rangle \geqslant 0 \quad \forall\ x \in D$$

i. e., System 2 has a solution (y_1^*, y_2^*). The proof is thus complete.

Definition 2.1 The set-valued map $f: D \to 2^{Y_1}$ is called nearly subconvexlike, if $\exists\ u \in int\ Y_{1+}$, $\exists\ \alpha \in (0, 1)$, $\forall\ x_1, x_2 \in D$, $\forall\ \varepsilon > 0$, $\exists\ z \in D$, such that

$$\varepsilon u + \alpha f(x_1) + (1-\alpha) f(x_2) \subset f(z) + Y_{1+}$$

Lemma 2.5 Let $M = f(D) + int\ Y_{1+}$. If the set-valued map f is nearly subconvexlike, then M is nearly convex.

Theorem 2.2 Suppose that the set-valued map $f: D \to 2^{Y_1}$ is nearly subconvexlike, and suppose that the interior of $f(D) + Y_{1+}$ is nonempty. Then, exactly one of the following statements is true:

1) $\exists\ x_0 \in D$, s. t. $-f(x_0) \cap int\ Y_{1+} \neq \varnothing$;

2) $\exists\ y_1^* \in Y_{1+}^*$, $y_1^* \neq O_{Y_1^*}$, s. t. $\langle f(D), y_1^*\rangle \geqslant 0$

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近类凸集值映射下的择一性定理

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摘要：在线性拓扑空间中，得到若干个近类凸、近次类凸集值映射下的择一性定理。

关键词：择一性定理；集值映射；近类凸；近次类凸；线性拓扑空间